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Realization of Kirillov–Reshetikhin crystals $B^{1,s}$ for $\widehat{\mathfrak{sl}}_n$ using Nakajima monomials

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Abstract. We give a realization of the Kirillov–Reshetikhin crystal $B^{1,s}$ for $\widehat{\mathfrak{sl}}_n$ using Nakajima monomials using the crystal structure given by Kashiwara. We describe the tensor product $\bigotimes_{i=1}^{N} B^{1,s_i}$ in terms of a shift of indices, allowing us to recover the Kyoto path model. We give a description of the limit of the coherent family of crystals $\{B^{1,s}\}_{s=1}^{\infty}$ using Nakajima monomials, which allows us to recover the path model for $B(\infty)$. Additionally, we realize the KR crystals $B^{r,1}$ for $\widehat{\mathfrak{sl}}_n$ using Nakajima monomials.

Keywords: crystal, Kirillov–Reshetikhin crystal, Nakajima monomial, quantum group

1 Introduction

A special class of finite-dimensional modules of the derived subalgebra Drinfel'd–Jimbo quantum group $U_q'(\widehat{\mathfrak{sl}}_n)$ called Kirillov–Reshetikhin (KR) modules have received significant attention over the past 20 years. KR modules have many remarkable properties and deep connections with mathematical physics. For example, KR modules arise in the study of certain solvable lattice models [15, 25]. Their characters (respectively q-characters [8, 9]) satisfy the Q-system (respectively T-system) relations, which come from a certain cluster algebra [12, 37]. This gives a fermionic formula interpretation and a relation to the string hypothesis in the Bethe ansatz for solving Heisenberg spin chains. The graded characters of (respectively Demazure submodules of) tensor products of certain KR modules, the fundamental representations, are (respectively nonsymmetric) Macdonald polynomials at $t = 0$ [32, 33] (respectively [35]).

In the seminal papers [21, 22], Kashiwara defined the crystal basis of a representation of a quantum group and that every irreducible highest weight representation admits a crystal basis $B(\lambda)$. While KR modules are cyclic modules, they are not highest weight modules. Yet, KR modules for $U_q'(\widehat{\mathfrak{sl}}_n)$ admit crystal bases [20], which are known as Kirillov–Reshetikhin (KR) crystals, and contain even further connections to mathematical physics. For example, KR crystals are in bijection with rigged configurations [4, 26, 27,

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28], combinatorial objects that arise naturally from the Bethe ansatz. KR crystals $B^{1,s}$ can be used to model box-ball systems [44]. KR crystals are perfect [7] and used in the Kyoto path model [19, 20, 41], which arose from the study of integrable 2D lattice models.

Despite intense study, relatively little is understood about KR crystals. In particular, there is currently not a combinatorial model for KR crystals where all crystal operators are given by the same rules, the model is valid for general $B^{r,s}$, and the model extends to all affine types. By using the decomposition into $U_q(\mathfrak{sl}_n)$-crystals and the Dynkin diagram automorphism, we can lift the tableaux model of [24] to a model for KR crystals for $U_q'(\widehat{\mathfrak{sl}}_n)$ [43], but at the cost of obscuring the e_0 and f_0 crystal operators.

Partial progress has been made on this problem. Naito and Sagaki constructed a model uniform across all types for $\bigotimes_{i=1}^N B^{r_i,1}$ by using the usual crystal structure on Lakshmibai–Seshadri (LS) paths and projecting onto the classical weight space, where an equivalent description is given by quantum LS paths (see [32, 34] and references therein). Lenart and Lubovsky constructed $\bigotimes_{i=1}^N B^{r_i,1}$ by using a discrete version of quantum LS paths called the quantum alcove path model [31]. Yet, it is not known how to extend these models for general $B^{r,s}$. On the other side, there are models for $B^{r,s}$ in type $A_n^{(1)}$, for example [30], but these are not known to extend (uniformly) to other affine types.

There is a t-analog of q-characters (or q,t-characters for short) that was studied by Nakajima [39, 37, 38, 40, 36]. From this study, a $U_q(\mathfrak{sl}_n)$-crystal structure on the monomials that appear in the q-character was given by Nakajima [38]. Kashiwara [23] independently constructed a different crystal structure on the q-character monomials. Both models were generalized by Sam and Tingley [42], who also made a connection to quiver varieties, which is known as the Nakajima monomial model.

Nakajima's q,t-characters have also been well-studied using a variety of techniques. While their definition is combinatorial, Nakajima used quiver varieties to show their existence [39]. Hernandez reformulated the definition to be purely algebraic by using a t-analog of screening operators [11]. Nakajima also showed that q,t-characters can be used to determine the change of basis from standard to simple $U_q(\widehat{\mathfrak{sl}}_n)$-modules in the Grothendieck group [36]. Kodera and Naoi then connected this to the graded decomposition into $U_q(\mathfrak{sl}_n)$-modules of a tensor product of fundamental representations [29], which give Kostka polynomials [28] and Macdonald polynomials at $t = 0$ [32, 33].

Cluster algebras [6] also have strong connections to characters of KR crystals and Nakajima monomials. Hernandez and Leclerc gave an algorithm to compute q-characters as certain cluster variables from a semi-infinite quiver [13]. Kanakubo and Nakashima showed that the generalized minors of the double Bruhat cell $G^{u,e} = BuB \cap B_-eB_-$ can be expressed as the sum over the Nakajima monomials in a Demazure subcrystal [16] and are the cluster variables of the coordinate ring $\mathbb{C}[G^{u,e}]$, an upper cluster algebra [1].

This is evidence that there should exist a natural description of tensor products of KR crystals in terms of Nakajima monomials. Hernandez and Nakajima [14] construct $\bigotimes_{i=1}^N B^{r_i,1}$ by a similar construction to the projected LS paths, and likewise, it does not

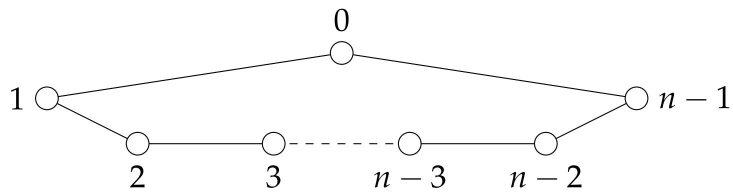

Figure 1: Dynkin diagram of $\widehat{\mathfrak{sl}}_n$.

construct higher level KR crystals. Our main result is a construction of $B^{1,s}$ for $U'_q(\widehat{\mathfrak{sl}}_n)$ using Nakajima monomials. From this construction, we are able to describe $\bigotimes_{i=1}^{N} B^{1,s_i}$ without using tensor products. Moreover, we recover the Kyoto path model. From this construction, we are able to relate the models of [42, 45] with the Kyoto path model. We also extend our construction to the coherent limit of $\{B^{1,s}\}_{s=1}^{\infty}$, where we recover the path model for $B(\infty)$ [17] and the isomorphism with Nakajima monomials given in [18].

Our results suggest a crystal interpretation for the fusion construction of [19, 20]. Indeed, the kernel of the R-matrix can be given by a (twisted) commutator relation on the elements of $B^{1,1}$. By considering the tensor product as multiplication, we can relate our construction with the kernel of the R-matrix. Furthermore, our construction gives an explanation of the link between the models explored in [42, 46, 45]. While our model does not naturally extend to general $B^{r,s}$ or to other affine types, these links are evidence that our construction can be modified to the general case.

This is an extended abstract of [10] and is organized as follows. In Section 2, we give the necessary background. In Section 3, we describe our main results. In Section 4, we describe $B^{r,1}$ using Nakajima monomials. In Section 5, we relate our construction with the kernel of the R-matrix.

2 Background

2.1 Crystals

Let $\widehat{\mathfrak{sl}}_n$ be the affine Kac–Moody Lie algebra of type $A_{n-1}^{(1)}$ with index set $I = \{0, 1, \dots, n-1\}$ (see Figure 1 for the Dynkin diagram), Cartan matrix $(a_{ij})_{i,j \in I}$, simple roots $\{\alpha_i\}_{i \in I}$, simple coroots $\{h_i\}_{i \in I}$, fundamental weights $\{\Lambda_i\}_{i \in I}$, weight lattice P, dual weight lattice $P^\vee$, canonical pairing $\langle\ ,\ \rangle \colon P^\vee \times P \to \mathbb{Z}$ given by $\langle h_i, \alpha_j \rangle = a_{ij}$, and quantum group $U_q(\widehat{\mathfrak{sl}}_n)$. Let P^+ denote the dominant integral weights. The level of $\lambda \in P$ is $\langle c, \lambda \rangle$, where $c = h_0 + h_1 + \cdots + h_{n-1}$ is the canonical central element of $\widehat{\mathfrak{sl}}_n$. Note that $\mathfrak{sl}_n$ is the canonical simple Lie algebra given by the index set $I_0 = I \setminus \{0\}$. Let $\{\overline{\Lambda}_i\}_{i \in I_0}$ denote the fundamental weights of $\mathfrak{sl}_n$.

We write $U_q'(\widehat{\mathfrak{sl}}_n) = U_q([\widehat{\mathfrak{sl}}_n, \widehat{\mathfrak{sl}}_n])$, and let $\delta = \alpha_0 + \alpha_1 \cdots + \alpha_{n-1}$ denote the null root. Note that the $U_q'(\widehat{\mathfrak{sl}}_n)$ fundamental weights and simple roots are also given by $\{\Lambda_i\}_{i \in I}$ and $\{\alpha_i\}_{i \in I}$, respectively, but are considered in the weight lattice $P/\mathbb{Z}\delta$.

An $U_q(\widehat{\mathfrak{sl}}_n)$-*crystal* is a set B together with *crystal operators* $e_a, f_a \colon B \longrightarrow B \sqcup \{0\}$, maps $\varepsilon_a, \varphi_a \colon B \longrightarrow \mathbb{Z} \sqcup \{-\infty\}$, and a *weight map* $\mathrm{wt} \colon B \longrightarrow P$ satisfying certain conditions (see, for example, [21, 22]). We say an element $b \in B$ is *highest weight* if $e_i b = 0$ for all $i \in I$.

Kashiwara showed in [22] that the irreducible highest weight $U_q(\widehat{\mathfrak{sl}}_n)$-module $V(\lambda)$ admits a crystal basis, where $\lambda \in P^+$. We denote this crystal basis by $B(\lambda)$, and let $u_\lambda \in B(\lambda)$ denote the unique highest weight element. The crystal corresponding to $U_q^-(\widehat{\mathfrak{sl}}_n)$, the lower half of $U_q(\widehat{\mathfrak{sl}}_n)$, is denoted by $B(\infty)$ with highest weight element u_∞.

We define the *tensor product* of abstract $U_q(\widehat{\mathfrak{sl}}_n)$-crystals B_1 and B_2 as the crystal $B_2 \otimes B_1$ that is the Cartesian product $B_2 \times B_1$ with the crystal structure

$$e_i(b_2 \otimes b_1) = \begin{cases} e_i b_2 \otimes b_1 & (\varepsilon_i(b_2) > \varphi_i(b_1)), \\ b_2 \otimes e_i b_1 & \text{otherwise}, \end{cases} \qquad f_i(b_2 \otimes b_1) = \begin{cases} f_i b_2 \otimes b_1 & (\varepsilon_i(b_2) \geq \varphi_i(b_1)), \\ b_2 \otimes f_i b_1 & \text{otherwise}, \end{cases}$$

$$\varepsilon_i(b_2 \otimes b_1) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \mathrm{wt}(b_1) \rangle),$$
$$\varphi_i(b_2 \otimes b_1) = \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, \mathrm{wt}(b_2) \rangle),$$
$$\mathrm{wt}(b_2 \otimes b_1) = \mathrm{wt}(b_2) + \mathrm{wt}(b_1).$$

Remark 2.1. *Our tensor product convention is opposite of Kashiwara [22].*

Let B_1 and B_2 be two $U_q(\widehat{\mathfrak{sl}}_n)$-crystals. A *crystal embedding* $\psi \colon B_1 \to B_2$ is an injection $B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ with $\psi(0) = 0$ such that, for all $i \in I$, we have $\psi(e_i b) = e_i \psi(b)$ and $\psi(f_i b) = f_i \psi(b)$ for all $b \in B_1$ and ε_i, φ_i, and wt are preserved under ψ. An *isomorphism* is a crystal embedding that is also a bijection.

2.2 Nakajima monomials

Next, we give the Nakajima monomial model as a special case of [23, 42].

Let $\mathcal{M}$ denote the set of Laurent monomials in the commuting variables $\{Y_{i,k}\}_{i \in I, k \in \mathbb{Z}}$. For a monomial $m = \prod_{i \in I} \prod_{k \in \mathbb{Z}} Y_{i,k}^{y_{i,k}}$, define

$$\varepsilon_i(m) = -\min_{k \in \mathbb{Z}} \sum_{s > k} y_{i,s}, \qquad k_e(m) = \max \left\{ k \;\middle|\; \varepsilon_i(m) = -\sum_{s > k} y_{i,s} \right\},$$

$$\varphi_i(m) = \max_{k \in \mathbb{Z}} \sum_{s \leq k} y_{i,s}, \qquad k_f(m) = \min \left\{ k \;\middle|\; \varphi_i(m) = \sum_{s \leq k} y_{i,s} \right\},$$

and $\operatorname{wt}(m) = \sum_{i \in I} \sum_{k \in \mathbb{Z}} y_{i,k} \Lambda_i$. Define the crystal operators $e_i, f_i \colon \mathcal{M} \to \mathcal{M} \sqcup \{0\}$ by

$$e_i(m) = \begin{cases} 0 & \text{if } \varepsilon_i(m) = 0, \\ m A_{i,k_e(m)-1} & \text{if } \varepsilon_i(m) > 0, \end{cases} \qquad f_i(m) = \begin{cases} 0 & \text{if } \varphi_i(m) = 0, \\ m A^{-1}_{i,k_f(m)-1} & \text{if } \varphi_i(m) > 0, \end{cases}$$

where $A_{i,k} = Y_{i,k} Y_{i,k+1} Y^{-1}_{i-1,k} Y^{-1}_{i+1,k+1}$. We also denote $Y_\lambda := \prod_{i \in I} Y_{i,0}^{\langle h_i, \lambda \rangle}$ and $\mathbf{1} = Y_0$.

Theorem 2.2 ([42]). *Let $\lambda \in P^+$, and let $\mathcal{M}(\lambda)$ denote the closure of Y_λ under e_i and f_i for all $i \in I$. Then we have $\mathcal{M}(\lambda) \cong B(\lambda)$ as $U_q'(\widehat{\mathfrak{sl}}_n)$-crystals.*

Next, we define the *modified crystal operators* e_i' and f_i' by instead using

$$k_e'(m) = \begin{cases} 0 & \text{if } k_e(m) \text{ is undefined}, \\ k_e(m) & \text{otherwise}, \end{cases} \qquad k_f'(m) = \begin{cases} 0 & \text{if } k_f(m) \text{ is undefined}, \\ k_f(m) & \text{otherwise}. \end{cases}$$

Theorem 2.3 ([18]). *Let $\mathcal{M}(\infty)$ denote the closure of $\mathbf{1}$ under e_i and f_i' for all $i \in I$. Then we have $\mathcal{M}(\infty) \cong B(\infty)$ as $U_q'(\widehat{\mathfrak{sl}}_n)$-crystals.*

2.3 Kirillov–Reshetikhin crystals and the Kyoto path model

A *Kirillov–Reshetikhin (KR) module* $W^{r,s}$, where $r \in I_0$ and $s \in \mathbb{Z}_{>0}$, is a particular irreducible finite-dimensional $U_q'(\widehat{\mathfrak{sl}}_n)$-module that has many remarkable properties. KR modules are classified by their Drinfel'd polynomials, and $W^{r,s}$ is the minimal affinizations of the highest weight $U_q(\mathfrak{sl}_n)$-representation $V(s\Lambda_r)$ [2, 3]. The module $W^{r,s}$ admits a crystal basis [20], which is denoted by $B^{r,s}$ and called a *Kirillov–Reshetikhin (KR) crystal*.

KR crystals have many important properties. One property is that the KR crystal $B^{r,s}$ is a *perfect crystal of level* s, a technical condition that we do not explicitly need here (see, *e.g.*, [20] for a precise definition). Another property is $B^{r,s} \cong B(s\overline{\Lambda}_r)$ as $U_q(\mathfrak{sl}_n)$-crystals.

We will be focusing on the KR crystal $B^{1,s}$, which is known to admit the following model from the *vector representation*. We have

$$B^{1,s} = \left\{ (x_1, \dots, x_n) \;\middle|\; x_1, \dots, x_n \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^n x_i = s \right\}$$

with the crystal structure

$$e_i(x_1, \dots, x_n) = \begin{cases} 0 & \text{if } x_{i+1} = 0, \\ (\dots, x_i + 1, x_{i+1} - 1, \dots) & \text{if } x_{i+1} > 0, \end{cases} \qquad \varepsilon_i(x_1, \dots, x_n) = x_{i+1},$$

$$f_i(x_1, \dots, x_n) = \begin{cases} 0 & \text{if } x_i = 0, \\ (\dots, x_i - 1, x_{i+1} + 1, \dots) & \text{if } x_i > 0, \end{cases} \qquad \varphi_i(x_1, \dots, x_n) = x_i,$$

and $\mathrm{wt}(x_1, \ldots, x_n) = \sum_{i \in I}(x_i - x_{i+1})\Lambda_i$, where all indices are understood mod n.

Theorem 2.4 ([19, 20, 41]). *Let $\lambda \in P^+$ be a level s weight. Let B be a perfect crystal of level s. Let $b^\lambda \in B$ be the unique element such that $\varphi(b^\lambda) = \lambda$. Let $\mu = \varepsilon(b^\lambda)$. Then the map $\Psi \colon B(\lambda) \to B \otimes B(\mu)$ defined by $u_\lambda \mapsto b^\lambda \otimes u_\mu$ is a $U_q(\widehat{\mathfrak{sl}}_n)$-crystal isomorphism.*

For a level s weight λ, we can construct a model for $B(\lambda)$ by iterating Ψ:

$$\Psi^{+\infty} \colon B(\lambda) \to B^{1,s} \otimes B^{1,s} \otimes \cdots \tag{2.5}$$

since $B^{1,s}$ is a perfect crystal of level s. This is the *Kyoto path model*. Furthermore, $\Psi^{+\infty}(u_\lambda)$ is eventually cyclic and, for any $b \in B(\lambda)$, $\Psi^{+\infty}(b)$ only differs from $\Psi^{+\infty}(u_\lambda)$ in a finite number of factors. Therefore, we can consider $\Psi^N(b)$ for $N \gg 1$ (that depends on b) to define the crystal structure on the Kyoto path model using only the KR crystal $B^{1,s}$.

There is also an analog of the Kyoto path model given in [17] for $B(\infty)$. We first need the *coherent limit* B_∞ of the family $\{B^{1,s}\}_{s=1}^{\infty}$, which is formed by taking the closure of $b_\infty = (0, 0, \ldots, 0)$ under the crystal operators $e_i(x_1, \ldots, x_n) = (\ldots, x_i + 1, x_{i+1} - 1, \ldots)$ and $f_i(x_1, \ldots, x_n) = (\ldots, x_i - 1, x_{i+1} + 1, \ldots)$ with $\mathrm{wt}(x_1, \ldots, x_n) = \sum_{i \in I}(x_i - x_{i+1})\Lambda_i$. Thus, we can construct $B(\infty)$ by iterating the following.

Theorem 2.6. *Let B_∞ denote the coherent limit of $\{B_s\}_{s=1}^{\infty}$, where B_s is a perfect crystal of level s. The map $\Omega \colon B(\infty) \to B_\infty \otimes B(\infty)$ given by $u_\infty \mapsto b_\infty \otimes u_\infty$ is a $U'_q(\widehat{\mathfrak{sl}}_n)$-crystal isomorphism.*

3 Nakajima monomial realization of $B^{1,s}$ and B_∞

In this section, we will state our main results. We first need to change our variables to $X_{i,k} := Y_{i-1,k+1}^{-1} Y_{i,k}$ introduced in [18]. Next, define

$$\mathcal{M}^{1,s} := \left\{ \prod_{i=1}^{n} X_{i,0}^{x_i} \;\middle|\; x_1, \ldots, x_n \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{n} x_i = s \right\}.$$

Theorem 3.1. *We have $B^{1,s} \cong \mathcal{M}^{1,s}$ as $U'_q(\widehat{\mathfrak{sl}}_n)$-crystals. Moreover, the isomorphism is given by $(x_1, \ldots, x_n) \mapsto X_{1,0}^{x_1} X_{2,0}^{x_2} \cdots X_{n,0}^{x_n}$.*

To define $\mathcal{M}^{1,s}$, we considered the crystal generated from $X_{1,0}^s$. However, by shifting the monomials, we can construct an isomorphism with the tensor product. Indeed, let τ_j be the (multiplicative) map given by $Y_{i,k} \mapsto Y_{i,k+j}$ for all $i \in I$ and $k \in \mathbb{Z}$. Let $\mathcal{M} \cdot \mathcal{M}' = \{m \cdot m' \mid m \in \mathcal{M}, m' \in \mathcal{M}'\}$ under the usual crystal operators.

Theorem 3.2. *For any sequence $0 < j_1 < \cdots < j_N$, we have $\prod_{k=1}^{N} \tau_{j_k}(\mathcal{M}^{1,s_k}) \cong \bigotimes_{k=1}^{N} B^{1,s_k}$ as $U'_q(\widehat{\mathfrak{sl}}_n)$-crystals.*

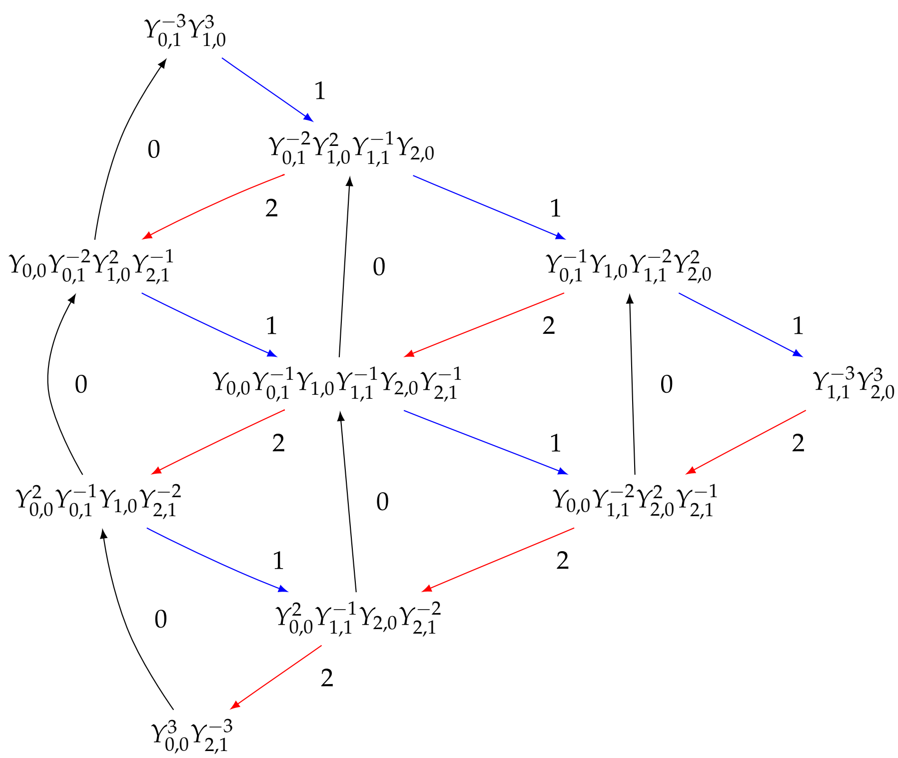

Figure 2: The crystal $\mathcal{M}^{1,3}$ for $\widehat{\mathfrak{sl}}_3$.

As a special case of Theorem 3.2, we have that $\bigotimes_{k=1}^{N} B^{1,s_k} \cong \prod_{k=1}^{N} \tau_{k-1}(\mathcal{M}^{1,s_k})$. Let $\Phi^{+\infty}$ be the isomorphism for $N \to \infty$. Note that $\Phi^{+\infty}$ recovers the Kyoto path model.

Theorem 3.3. *Let $\lambda \in P^+$ be a level s weight. Let $\Xi\colon B(\lambda) \to \mathcal{M}(\lambda)$ be the canonical isomorphism and $\Psi^{+\infty}\colon B(\lambda) \to B^{1,s} \otimes B^{1,s} \otimes \cdots$ from Equation (2.5). Then $\Phi^{+\infty} \circ \Psi^{+\infty} = \Xi$.*

Example 3.4. *Consider $U_q'(\widehat{\mathfrak{sl}}_5)$. Then we have*

$$\begin{aligned}
\Phi^{+\infty}(u_{\Lambda_0}) &= Y_{0,0}Y_{4,1}^{-1}\tau_1(Y_{4,0}Y_{3,1}^{-1})\tau_2(Y_{3,0}Y_{2,1}^{-1})\tau_3(Y_{2,0}Y_{1,1}^{-1})\tau_4(Y_{1,0}Y_{0,1}^{-1})\tau_5(Y_{0,0}Y_{4,1}^{-1})\cdots \\
&= Y_{0,0}Y_{4,1}^{-1}Y_{4,1}Y_{3,2}^{-1}Y_{3,2}Y_{2,3}^{-1}Y_{2,3}Y_{1,4}^{-1}Y_{1,4}Y_{0,5}^{-1}Y_{0,5}Y_{4,6}^{-1}\cdots \\
&= Y_{0,0}
\end{aligned}$$

We can also modify our construction to describe B_∞ in terms of Nakajima monomials. Define $\mathcal{M}_\infty$ as the closure of **1** under the modified crystal operators e_i' and f_i'. We recover the path model for $B(\infty)$ and [18, Thm. 5.1].

Theorem 3.5. *We have $B_\infty \cong \mathcal{M}_\infty$ as $U_q'(\widehat{\mathfrak{sl}}_n)$-crystals, where the isomorphism is given by $(x_1, \dots, x_n) \mapsto X_{1,0}^{x_1} \cdots X_{n,0}^{x_n}$. Moreover, the map $\Theta\colon \mathcal{M}_\infty \otimes \mathcal{M}(\infty) \to \mathcal{M}(\infty)$ given by $m \otimes m' \mapsto m \cdot \tau_1(m')$ is a $U_q'(\widehat{\mathfrak{sl}}_n)$-crystal isomorphism.*

4 Nakajima monomial realization of $B^{r,1}$

The KR crystal $B^{r,1}$ can be described by $(x_1, \dots, x_n)$ such that $0 \le x_i \le 1$, for all $1 \le i \le n$, and $\sum_{i=1}^{n} x_i = r$ with the crystal structure the same as the vector representation given in Section 2.3. Hence, we have the following fact, which does not appear in the literature as far as we are aware, but is likely known to experts.

Proposition 4.1. *There exists a crystal embedding $B^{r,1} \to B^{1,r}$.*

Therefore, we can quotient our monomials by $X_{i,k}^2$ and obtain a description for $B^{r,1}$ in terms of Nakajima monomials. We could also consider $X_{i,k}$ as anticommuting variables (up to sign) since $V(\overline{\Lambda}_r)$ can be constructed from $\bigwedge^r V(\overline{\Lambda}_1)$. Explicitly, given a monomial m, define modified crystal operators $\overline{f}_i$ by $\overline{f}_i(m) = f_i(m)$ if $f_i(m)$ does not contain an $X_{i,k}^2$ for some $(i,k) \in I \times \mathbb{Z}$ and $\overline{f}_i(m) = 0$ otherwise. The definition of $\overline{e}_i$ is defined similarly by replacing f_i with e_i. Let $\overline{\mathcal{M}}(m)$ denote the closure of m under $\overline{e}_i$ and $\overline{f}_i$.

Proposition 4.2. *For any $k \in \mathbb{Z}$, we have $B^{r,1} \cong \overline{\mathcal{M}}\left(\prod_{i=1}^{r} X_{i,k}\right)$ as $U_q'(\widehat{\mathfrak{sl}}_n)$-crystals.*

5 Relation to the R-matrix

From Theorem 3.2, we can construct the tensor product $B^{1,1} \otimes B^{1,1}$ by considering the product $\mathcal{M}^{1,1}$ with its shifted version $\tau_1(\mathcal{M}^{1,1})$. However, if we want to avoid the shift, we can still construct $\mathcal{M}^{1,1} \otimes \mathcal{M}^{1,1}$ by multiplication but having multiplication twisted by a generic parameter t. This is a special case of the results of [11, 29, 39, 37, 38] given in terms of Kashiwara's variation of Nakajima monomials. The kernel of the appropriate R-matrix is generated by $K = \{m \otimes m' - tm' \otimes m \mid m \neq m' \in \mathcal{M}^{1,1}\}$, which are twisted commutators. Thus, the quotient $(\mathcal{M}^{1,1})^{\otimes 2}/K \cong \mathcal{M}^{1,2}$, and this can be extended to $\mathcal{M}^{1,s}$.

Example 5.1. *Consider the $U_q'(\widehat{\mathfrak{sl}}_3)$-crystals $\mathcal{M}^{1,1} \otimes \mathcal{M}^{1,1}$ and $\mathcal{M}^{1,2}$ (see Figure 3). The graded q-character of $\mathcal{T} = \mathcal{M}^{1,1} \otimes \mathcal{M}^{1,1}$ is*

$$\sum_{m \otimes m' \in \mathcal{T}} t^{E(m \otimes m')} m \cdot m' = Y_{0,1}^{-2} Y_{1,0}^{2} + (t+1)\, Y_{0,0} Y_{1,1}^{-1} Y_{2,0} Y_{2,1}^{-1} + (t+1)\, Y_{0,0} Y_{0,1}^{-1} Y_{1,0} Y_{2,1}^{-1}$$
$$+ (t+1)\, Y_{0,1}^{-1} Y_{1,0} Y_{1,1}^{-1} Y_{2,0} + Y_{1,1}^{-2} Y_{2,0}^{2} + Y_{0,0}^{2} Y_{2,1}^{-2},$$

where $E(b)$ is the energy of b [19, 20]. By considering the (graded) decomposition into $U_q(\mathfrak{sl}_3)$-crystals, we get the same decomposition (after $t \mapsto t^2$) as computed by [11, 39, 37, 38] (note that the Nakajima monomials we use are different). This correspondence is an example of the results of [29]. If we quotient by K, we recover the crystal graph of $\mathcal{M}^{1,2}$.

Due to the kernel of the R-matrix being essentially a twisted commutator, we believe our construction for $B^{1,s}$ only works for type $\widehat{\mathfrak{sl}}_n$. However, we expect a similar construction to work for the general case of $B^{r,s}$ and for all affine types.

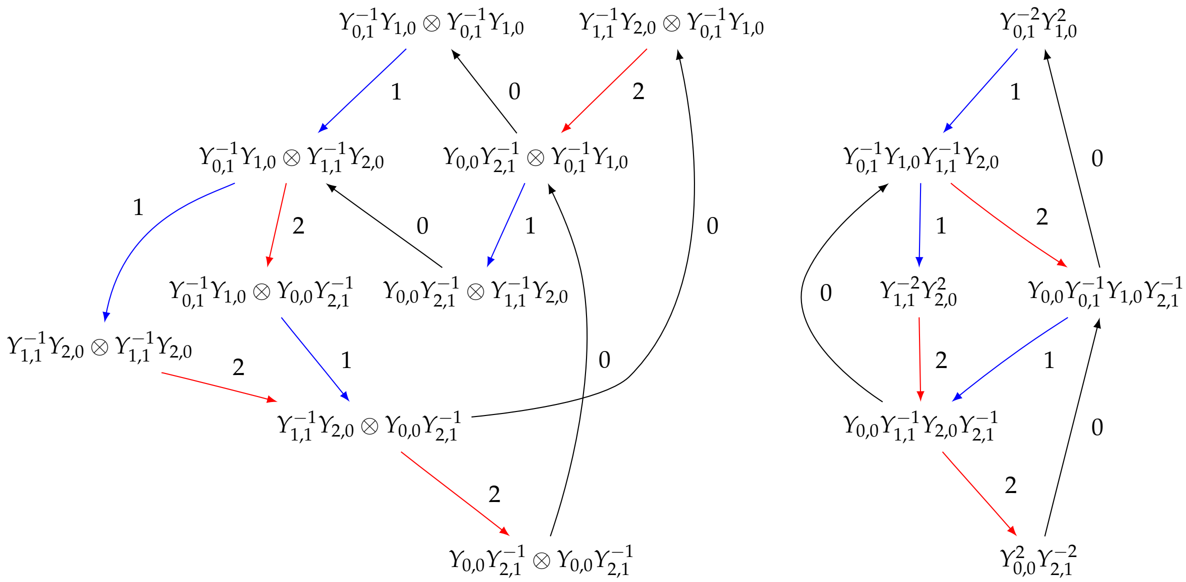

Figure 3: The $U_q'(\widehat{\mathfrak{sl}}_3)$-crystal $\mathcal{M}^{1,1} \otimes \mathcal{M}^{1,1}$ (left) and $\mathcal{M}^{1,2}$ (right).

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